The concept of real number, following Karl Weierstraß*

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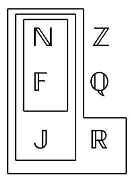
Abstract. A hitherto unknown construction of the real numbers is given. It was a unique idea by KARL WEIERSTRASS and it is the most elementary ever known.

Conway's ideal



> "Proceed from the natural numbers to the non-negative rationals (or the strictly positive ones if you prefer), then construct the non-negative (or positive) reals from these, so having no sign-problem, and then construct signed reals from these in the way that we constructed the signed integers from the natural numbers. I think that this is in fact the simplest way to construct the real numbers along traditional lines.
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> [...] We start from what is essentially just set theory, given half-a-dozen inductive definitions and a dozen inductive proofs (mostly one-liners), and we have a field of all surreal numbers. Since this includes the smaller field of all real numbers, we surely have a much simpler construction of these than the traditional one?"[1]

As is known, JOHN HORTON CONWAY (1937–2020) did not follow the above ideal but constructed his positive and negative numbers together. However, until now it has not been realized that a century earlier KARL WEIERSTRASS (1815–97) already proceeded along the very same lines suggested by CONWAY. WEIERSTRASS did not even need the fractions to construct his irrational and his real numbers, only to order them.

1 Natural, fractional and irrational quantities

Let us take the natural numbers $\mathbb{N}$ be given, $0 \notin \mathbb{N}$.

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[1] Conway 1994, pp. 95, 99 (only the text)

BASIC DEFINITIONS. Call each $\mathbb{N} \xrightarrow{f} \mathbb{N} \cup \{0\}$ a *quantity,* and distinguish:

1. if $f(1) \neq 0$ and $f(n) = 0$ otherwise, we have a *natural quantity,*
2. if $|\, f^{-1}(\mathbb{N} \setminus 0)\,|$ is finite, we have a *fractional quantity,* and
3. all others are called *irrational quantities.*

The *idea* is, of course, to think of an element $(n, f(n))$ as a usual fraction $\frac{f(n)}{n}$. Then f is a set of usual fractions, no two of them having the same denominator. A fractional quantity is a ruler with finite precision, an irrational quantity is a ruler with infinite precision.

EXAMPLES.

$$\left\{\tfrac{3}{1}, \tfrac{9}{3}\right\} \quad \text{is a fractional quantity},$$
$$\left\{\tfrac{3}{10^1}, \tfrac{3}{10^2}, \tfrac{3}{10^3}, \dots\right\} \quad \text{is an irrational quantity.}$$

DEFINITIONS.

1. The *sum* of two quantities is defined pointwise:

$$(f+g)(n) := f(n) + g(n)\,.$$

2. The *product* of two quantities is their convolution:

$$(f \cdot g)(k) \;=\; \sum_{n \cdot m = k} f(n) \cdot g(m)\,.$$ [2]

REMARKS.

(a) Addition and multiplication are *associative* as well as *commutative,* and the multiplication is *distributive* against the addition. (These features are hereditary from $\mathbb{N}$ and the set-functions to the quantities.)

(b) Addition, as well as multiplication, is in general *not reversible,* because set-operations are not.

DEFINITION.

3. A fractional quantity f is called at *constituent* of a quantity g, if there exist finitely many (additive) transformations of the elements of g such that the result g' satisfies:

$$f \subset g'\,.$$

(We know what an additive *transformation* of a usual fraction is.)

EXAMPLE. $\left\{\frac{1}{3}\right\}$ is a constituent of $\left\{\frac{1}{2}\right\}$, because

$$\tfrac{1}{2} = \tfrac{3}{6} = \tfrac{2}{6} + \tfrac{1}{6} = \tfrac{1}{3} + \tfrac{1}{6}\,.$$

[2] These formulations of the matter were invented by my admired teacher Prof. KLAUS KEIMEL (1939–2017) in August 2017.

DEFINITION.

4. Two (irrational) quantities are *equal* (=), if each constituent of one of them is a constituent of the other quantity.

EXAMPLE.
$$\left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3} \dots\right\} = \left\{\frac{1}{1}\right\}.$$

This proof demands analytical arguments.

Having fixed equality, we have to prove:

THEOREM. *If $g = g'$, we have $f \circ g = f \circ g'$ ("$\circ$" means "$+$" or "$\cdot$").*

This can be shown as g' is derived from g by finitely many transformations.

DEFINITIONS.

5. If in definition 4 only one case holds, this defines the relation *less than* ($<$).
6. An irrational quantity q is called *finite,* if there exists a natural quantity n such that $q < n$.

(The last definition equivocally gives the notion *convergence* for series of usual fractions.)

2 Real quantities

We recall Conway 1994: "[...] and then construct signed reals from [the positive reals] in the way that we constructed the signed integers from the natural numbers. I think that this is in fact the simplest way to construct the real numbers along traditional lines." KARL WEIERSTRASS was indeed able to proceed in this way, for today we can say:

The set of all (finite) quantities, together with 0*, is an additive—as well as a multiplicative—monoid.*

We are used to expanding a monoid H to a group $H \times H$ and taking care about ordered pairs. However, as in the case of complex numbers (we often take $x + iy$ instead of (x, y)), the order is arbitrary; we only have to *discern* between the *one* component ("real") and the *other* one ("imaginary"). Therefore, we create (like WEIERSTRASS) the

DEFINITION. A *real quantity*$_-$ a is an unequal pair

$$a := q + h\,r\,,$$

where q and r are finite irrational quantities or 0 and h ($\neq 1$) a symbol with the meaning $h^2 = 1$.

Caution: *A real quantity*$_-$ *is not a quantity!*

These real quantities$_-$ $q + h\,r$, together with today's familiar definitions

$$\begin{aligned} q + h\,r = s + h\,t \quad &:\Longleftrightarrow \quad q + t = s + r \\ (q + h\,r) \oplus (s + h\,t) \quad &:= \quad (q + s) + h(r + t) \\ (q + h\,r) \ominus (s + h\,t) \quad &:= \quad (q + t) + h(s + r) \end{aligned}$$

form an additive group and together with the trivial definition

$$(q + h\,r) \odot (s + h\,t) \quad := \quad (q \cdot s + r \cdot t) + h(q \cdot s + t \cdot r)$$

a ring with 1.

We are yet to define
(i) *true* (not only *formal*) *subtraction* and
(ii) *division.*

3 Values

A quantity is a function $\mathbb{N} \xrightarrow{f} \mathbb{N} \cup \{0\}$. Going further than WEIERSTRASS, we add two conditions:

1. $\operatorname{dom} f = \left\{ z^k \mid k \in \mathbb{N} \cup \{0\} \right\}$ for a base $z \in \mathbb{N} \setminus \{1\}$; or $\operatorname{dom} f = \{ k! \mid k \in \mathbb{N} \}$; or the like.
2. $\frac{f(n)}{n} < \frac{1}{m}$, where $f(m) \neq 0$ and $f(k) = 0$ for $n < k < m$,

and we demand the

> DEFINITION. A *place value number* (with base $z \in \mathbb{N} \setminus \{1\}$) is a quantity, which meets both conditions above.

We know that decimal numbers and binary numbers are place value numbers. Now we have the following important

> THEOREM. *For each finite quantity q and any place value system P we have a place value number $q^{(P)}$ with $q = q^{(P)}$.*

PROOF (for base $z = 2$). Let q be some finite quantity > 0. Obviously, there exists a fractional quantity f with $0 < f < q$. But we can transform a fractional quantity f to a usual fraction $f_u = f$ and this number to a place value number $p_f = f_u$. Therefore, we obtain a place value number p_f with $0 < p_f < q$. Now, Hilbert 1923 proved that the binary numbers are *complete.* That is why there exists the place value number

$$q^{(2)} := \sup \left\{ p \mid p \text{ is a place value number with base 2 and } p < q \right\},$$

and we have $q = q^{(2)}$. ■

The mapping $q \mapsto q^{(P)}$ may be called a *representation* of q. Caution: *Representations* are not isomorphisms but only homomorphisms concerning addition and multiplication.

The fractional quantity f in the proof above does not depend on the place value system. Therefore, this proof shows the natural fact:

THEOREM. *All place value systems contain the same numbers.*

DEFINITIONS.

1. The quantity

   $$q^{(P)} \equiv W_z(q) := \sup \{\, p \mid p \text{ is a place value number (with base } z\text{) and } p < q \,\}$$

   is called the *representation* of the quantity q in the place value system P (with base z).

2. *Value of a quantity q* is its representation $q^{(P)} \equiv W_z(q)$ in a place value system P (with base z).

   EXAMPLES.
   $$\begin{aligned} W_{10}\left(\{\tfrac{2}{1}, \tfrac{3}{2}\}\right) &= 3.5\,, \\ W_{10}\left(\{\tfrac{3}{10^1}, \tfrac{3}{10^2}, \tfrac{3}{10^3}, \ldots\}\right) &= 0.\bar{3}\ldots \end{aligned}$$

3. *Value of a real quantity$_-$* $q + h\,r$ is its represenation $(q + h\,r)^{(P)} \equiv W_z(q + h\,r) := q^{(P)} - r^{(P)} \equiv W_z(q) - W_z(r)$ in a place value system P (with base z).

EXAMPLE.
$$W_{10}\left(\{\tfrac{2}{1}, \tfrac{3}{2}\}\,, \{\tfrac{3}{10^1}, \tfrac{3}{10^2}, \tfrac{3}{10^3}, \ldots\}\right) = 3.5 - 0.\bar{3}\ldots = 3.1\bar{6}\ldots$$

For centuries we have known how to subtract and how to divide place value numbers.[3] WEIERSTRASS' idea shows:

> *The usual place value numbers may be constructed without assuming any analytical fact.*

Handling place value numbers calls for some basic facts of calculus, a fact which is often not appreciated. In general, division demands such facts, and this shows: WEIERSTRASS' construction constitutes them as a ring, not a field. But the real numbers are just the real numbers—whatever you might think about their *structure*.[4]

The only analytical concept used in this construction (besides the *infinite set*) is the *supremum*. It is needed to prove the existence of the *value* of any finite irrational or real quantity$_{(-)}$.

[3] A modern textbook is Rautenberg 2007.

[4] "A set of terms has all the orders of which it is capable." and "just as we may notice among the fixed stars either their order of brightness or their distribution in the sky, so there are various relations among numbers which may be observed, and which give rise to various different orders among numbers, all equally legitimate." (Russell 1920, pp. 35 f.)

Philosophical aspects

At least since ARISTOTLE, traditional philosophy creates its concepts as *abstractions* from reality. For millenia mankind lived without negative numbers, just imagine the abacus. There is no real thing that might be abstrated to the number -1. Until today, WEIERSTRASS is the only mathematician who constructed the real numbers without resort to negative numbers.

Furthermore, WEIERSTRASS was the first (and is, up to my knowledge, till today the only) mathematician, who constructed the real numbers with unordered sets. Having known this, BERTRAND RUSSELL never would have claimed: "The generalizations of number—with the exception of the introduction of imaginaries, which must be independently effected—are all necessary consequences of the admission that the natural numbers form a progression."[5]

Some historiographical highlights

In WEIERSTRASS' thought, a quantity is a set, the elements of which are units e_n and their multiplicities k_n, *i.e.* $k_n \cdot e_n$; the *units* e_n are the "exact parts" of the *main unit* $e_1 := 1$.

It is absolute sensational that WEIERSTRASS (working earlier that 1887) founded his concept of real quantity on the concept of *set.* (It is well-known that GEORG CANTOR only coined the concept of set in 1895[6].) As WEIERSTRASS was not absolutely clear about this point, nobody understood this subtlety[7], and everybody thought his quantities were *sequences*—*cf. e.g.* BERTRAND RUSSELL[8] or JULES MOLK[9].

MOLK attended WEIERSTRASS' lecture in winter 1882/83[10]. So he improved ALFRED PRINGSHEIM's article on irrational numbers[11] in the *Encyklopädie der mathematischen Wissenschaften mit Einschluß ihrer Anwendungen* from 1898 in its french edition *Encyclopédie des Sciences Mathématiques Pures et Appliquées.*

The first sentence of MOLK's article reads:

> "In his lessons, Professor K. WEIERSTRASS at the university of Berlin called *numerical quantity* each 'ensemble', whose elements as well as the number, how often each element appears, were given."[12]

Now, in 1906 the french word "ensemble" meant: "whole", "sum", "unity", "correspondence", "cast", "interplay"—whatever, but it did not mean "set"; and it did not read "aggregate". But when the contemporary mathematician CHRISTOPHER TWEDDLE once came across MOLK's article[13], he wrongly read MOLK's "ensemble" as "set" and felt oblidged to for-

[5] Russell 1903, § 264, p. 278
[6] Cantor 1895–1897
[7] including myself: Spalt 1991; *cf.* Dieudonné 1992
[8] Russell 1903, ch. XXXIV, n. 268, p. 282
[9] Pringsheim and Molk 1904–1909, p. 152, footnote 56
[10] Vogt 1914, p. 381
[11] Pringsheim 1898
[12] Pringsheim and Molk 1904–1909, pp. 149 f.
[13] Tweddle 2011, p. 50, l. 10

mulate a manuscript entitled ”Weierstrass's Construction of the Irrational Numbers“. He did not cite MOLK,[14] when he pretended that WEIERSTRASS had given the definition:

> ”A *positive real number* is an equivalence class of aggregates which have finite value.“[15]

TWEDDLE did ”not rigorously verify the ordered field axioms are satisfied for WEIERSTRASS's theory of real numbers“[15], but he proved by induction (*sic!*):

> ”Lemma. *Each positive real number contains in its equivalence class a 'standard infinite decimal' representative.*“[16]

Unfortunately, the *Mathematische Semesterberichte* refused my manuscript in July 2019, wherein I pointed out TWEDDLE's misunderstandings and failures.

Now, how have I been able to correct the age-old misapprehension that WEIERSTRASS defined an irrational quantity to be a sequence?

The story is this: In summer 2016 an old filing cabinet in the *Mathematical Library* of the institute of mathematics of the *Goethe-University* of Frankurt was opened, and professor JÜRGEN WOLFART informed me about the discovery of an apparently interesting manuscript in August. This manuscript was written by EMIL STRAUSS, who attended WEIERSTRASS' lecture in winter 1880/81. It embraces 171 pages, the first 113 of which are concerned with the concepts of number up to and including the reals. After some months of thinking I decided to take the following two sentences word-for-word:

> ”Just for the record: two different aggregates can be considered as equal only if, first, both contain the same units, and second, each single unit is equally often contained in both of them. If we imagine the aggregate as a series of different things, then the succession of the different things is something unimportant to us, *i.e.* if we change the succession [or: sequence] of these things, there always remains a quantity, which we regard as equal to the former one.“[17]

If this were to be taken literally, WEIERSTRASS based his construction absolutely on sets, not on sequences. And the proof of the pudding is in the eating thereof.

I am indebted to JOHN D. SMITH, London, for correcting my language.

[14] Indeed, the name ”Molk“ is totally absent in Tweddle 2011.

[15] Tweddle 2011, p. 54

[16] Tweddle 2011, pp. 54 ff.

[17] Strauß (Winter 1880/81), pp. 30 f.—Later on, I discovered the sentence: ”A quantity, which is composed from infinitely many elements of the kind so far regarded, is conceptually completely defined, if it is possible to specify for every unit, how often it is contained in it.“—Kneser (Winter 1880/81), p. 23. (This sentence is even cited in the historical literature—see Purkert and Ilgauds 1987, p. 38—, but without being taken seriously.)—The third known lecture notes from winter 1880/81, Ramsay (Winter 1880/81), do not contain such a definition.

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